§7. Building Data – Part 1
(Decomposition of the branch locus of an Abelian cover)

Setting

- X connected normal variety, $X^\circ = X \setminus \mathrm{Sing}(X)$ with $\mathrm{codim}(\mathrm{Sing}(X)) \geq 2$
  Rem: (1) X° is still connected;
  (2) X° is dense on X.

  Y smooth complete algebraic variety.
- $\pi : X \longrightarrow Y$ finite morphism
- The degree of π is $\deg(\pi) := |\pi^{-1}(q)|$, q not a branch point of π.
- We consider the set $\mathrm{supp}(R) := \{ p \in X \mid \pi \text{ ramifies at } p \}$
  (namely $\det(d\pi_p) = 0$)

Remark Since Y is smooth, then $\mathrm{Sing}(X) \subseteq \mathrm{supp}(R)$.

- **Theorem (Zariski Purity)**
  $\mathrm{supp}(R)$ is always pure of codimension 1.

- **Def** R is the reduced (Weil) divisor of $\mathrm{supp}(R)$,
  namely $R = \sum_1^{\cdot} R_J$, R_J irreducible codim. 1 component of $\mathrm{supp}(R)$, with $i \neq J \Rightarrow R_i \neq R_J$

There is a unique Reduced Divisor $D = \sum_1^{\cdot} D_i \in \mathrm{Div}(Y)$ s.t.
$\mathrm{supp}(D) = \pi(\mathrm{supp}\, R)$

Remark Clearly $D \leq \pi_* R$ but in general $D \neq \pi_* R$.
As an example, take Example 2 with $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. In that

case $R = [1,0] + [0,1] + [i,1] + [i,-1] +$
$[1,1] + [1,-1]$
$= R_{e_2} + R_{e_1+e_2} + R_{e_1}$

$D = [1,0] + [2,-1] + [2,1]$
$= D_{e_2} + D_{e_1+e_2} + D_{e_1}$

$\Rightarrow \quad \pi^* D = 2R$ (since the ramific. index of each point is 2)

so $\quad 4D \underset{\text{projection formula}}{=} \pi_* \pi^* D = 2\pi_* R \Rightarrow \pi_* R = 2D \neq D.$

Let us consider a Galois cover $\pi : X \to Y$ with group G. Then

(1) $\forall g \in G, \quad \pi = \pi \circ g$

(2) $\deg(\pi) = |G|$

Lemma 1 $\quad p \notin \operatorname{supp}(R) \iff \underset{\{g \in G \mid g \cdot p = p\}}{\underset{\shortparallel}{\operatorname{Stab}_G(p)}} = 1_G$

proof $(\Rightarrow)$ $p \notin \operatorname{supp}(R) \Rightarrow p \in X^\circ$ and $d\pi_p$ is invertible $\Rightarrow$ by Inverse function thm. it there exists an open neigh. $\mathcal{U}$ of p s.t. $\pi_{|\mathcal{U}} : \mathcal{U} \to \pi(\mathcal{U})$ is an isom.

Let $g \in \operatorname{Stab}_G(p)$, we can define $\tilde{\mathcal{U}} := \bigcap_{k=1}^{o(g)} g^k \mathcal{U} \subseteq \mathcal{U}$.

Then $\pi_{|\tilde{\mathcal{U}}} : \tilde{\mathcal{U}} \to \pi(\tilde{\mathcal{U}})$ is an iso, so given $x \in \tilde{\mathcal{U}}$, then $\pi(gx) = \pi(x)$ but π is iso $\Rightarrow$ $gx = x \quad \forall x \in \tilde{\mathcal{U}} \Rightarrow g : X^\circ \to X^\circ$ is an holomorph. map trivial on $\tilde{\mathcal{U}} \underset{\text{(Identity thm, } X^\circ \text{ is connected)}}{\Rightarrow} g = Id$ on $X^\circ \Rightarrow$

g is the Id on X (X° is dense in X).

($\Leftarrow$) It there exists an open neighborhood U of p s.t. $U \cap gU = \phi$ $\forall g \in G$. But then $\pi_{|U} : U \to \pi(U)$ is an isomorphism $\Rightarrow$ $d\pi_p$ is invertible and $p \in X^\circ$. Thus, $p \notin \mathrm{supp}(R)$ ▨

Remark $\mathrm{Stab}_G(p)$ and $\mathrm{Stab}_G(g \cdot p)$ are conjugated to each other:

$$\boxed{\mathrm{Stab}_G(g \cdot p) = g \cdot \mathrm{Stab}_G(p) \cdot g^{-1}}$$

Thus

(1) If G is abelian, then $\mathrm{Stab}_G(g \cdot p) = \mathrm{Stab}_G(p)$;

(2) G sends points with a nontrivial stab to points with a nontrivial stab. In other words: G acts on supp(R), which is then a union of orbits.

Def Let T be an irreducible component of R. The Inertia Group of T is

$$In(T) := \{ g \in G \mid g \cdot p = p \ \forall p \in T \}$$

Remark Given $p \in T$, then $\mathrm{Stab}_G(p) \supseteq In(T)$.

Lemma 2 There is an open Zariski subset T° of T such that $\forall p \in T^\circ$ $\mathrm{Stab}(p) = \mathrm{In}(T)$.

proof We consider for any $g \in G$

$$\mathrm{Fix}(g) := \{ p \in X \mid g \cdot p = p \},$$

which is clearly a closed Zariski subset of X.

Then, $\bigcup_{g \in G \setminus \mathrm{In}(T)} \mathrm{Fix}(g)$ is still Zariski closed as G is finite

$\Rightarrow$ $T^\circ := T \setminus \bigcup_{g \in G \setminus \mathrm{In}(T)} \mathrm{Fix}(g)$ is the required open Zariski set. $\blacksquare$

Example It is not always true that

$$\mathrm{Stab}_G(p) = \mathrm{In}(T) \quad \forall p \in T$$

Example 4 of lecture 1 is a counterexample:

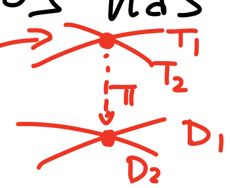
$X := \mathbb{P}^2$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle e_1, e_2 \rangle$

and $e_1 : X \to X$, $[x_0, x_1, x_2] \mapsto [x_0, -x_1, x_2]$, $e_2 : X \to X$, $[x_0, x_1, x_2] \mapsto [x_0, x_1, -x_2]$

The quotient map is $X \longrightarrow Y := \mathbb{P}^2(y_0, y_1, y_2)$, $[x_0, x_1, x_2] \mapsto [x_0^2, x_1^2, x_2^2]$

The ramification locus is $T_0 = (x_0 = 0)$, $T_1 = (x_1 = 0)$, $T_2 = (x_2 = 0)$. However, $p \in T_1$, $p = [1:0:0]$ has $\mathrm{Stab}(p) = \mathbb{Z}_2 \times \mathbb{Z}_2 \neq \mathrm{In}(T_1) = \langle e_1 \rangle$

The point has a bigger Stab.



Without lost of generality, we may assume

(1) T° is smooth;

(2) $T^\circ \subset X^\circ$

Theorem

1) $In(T)$ is a cyclic group;

2) If there exists a neighborhood $\mathcal{U}$ of a general point $p \in T$ with coordinates $t, z_2, \dots z_n$ such that

   (a) $T = (t=0)$

   (b) $In(T)$ acts as $(t, z_2, \dots z_n) \xmapsto{g} (\hat{\chi}(g)t, z_2, \dots z_n)$ for some character $\hat{\chi}$ s.t. $In(T)^* = \langle \hat{\chi} \rangle$.

   (c) $\hat{\chi}$ does not dep. on p neither on the local coord. but only on T.

   (d) $(t, z_2, \dots z_n) \xmapsto{\pi} (t^{|In(T)|}, z_2, \dots z_n)$

3) the ramification index of T for π is $|In(T)|$

Proof We choose $p \in T^\circ \Rightarrow \exists$ local coordinates $t, z_2, \dots z_n$ in a neighborhood $\mathcal{U}$ of p such that $\mathcal{U} \cap T^\circ = (t=0)$.

We can replace $\mathcal{U}$ by a restriction of $\bigcap_{g \in In(T)} g \cdot \mathcal{U}$ so that $g\,\mathcal{U} = \mathcal{U}$ $\forall g \in In(T)$ and $h\,\mathcal{U} \cap \mathcal{U} = \emptyset$ $\forall h \notin In(T)$.

$$\begin{tikzcd} \mathcal{U} \arrow[r, "g^{-1}"] \arrow[rrr, bend right, "g^*t"'] & \mathcal{U} \arrow[r, "{(t, z_2, \dots z_n)}"] & \mathbb{C}^n \arrow[r, "p_1"] & \mathbb{C} \end{tikzcd}$$

1-st projection

(clearly $h^* g^* t = (hg)^* t$)

In a similar way we define $g^* z_j$ using j-th project.

However, $g|_T = Id_T$, which forces

$$g^* z_J = z_J \mod t$$
$$g^* t = t u^g \quad \text{for some } u^g : \mathcal{U} \to \mathbb{C}$$

Indeed, we take the Taylor expansion at p (fixing $z_2, \dots z_n$)

$$g^* z_J = g_0^J(z_2, \dots z_n) + t g_1^J(z_2, \dots z_n) + t^2 g_2 + \cdots$$

but $g^* z_J|_T = z_J = g_0^J(0, z_2, \dots z_n) \Rightarrow g_0^J(z_2, \dots z_n) = z_J$

so $$g^* z_J = z_J + t g^J(t, z_2, \dots z_n) \equiv z_J \mod t$$

Instead, let us take the Taylor expansion at p of u^g: $\quad u^g = u_0^g + t u_1^g + \cdots = \sum t^k u_k^g(z_J)$

Then $u^g_{|T} = u_0^g$.

We observe that given $h, g \in In(T)$, then

$$(\text{mod } t^2) \quad h^* \underbrace{g^* t}_{t u^g} = g^* t(h^* t, h^* z_2, \dots h^* z_n) =$$
$$= (h^* t) \cdot u^g(h^* t, h^* z_2, \dots h^* z_n)$$
$$= t(u_0^h + t u_1^h + \cdots)(\underbrace{u_0^g(h^* z_2, \dots h^* z_n)} + t u_1^g(h^* z_2, \dots h^* z_n) + \cdots)$$
$$= t u_0^h \cdot u_0^g$$

Taylor exp. at p: $u_0^g(h^* z_2, \dots h^* z_n) = u_0^g(z_2, \dots z_n) +$ pieces of higher ord in t.

$$\Rightarrow h^* g^* t = t u_0^h u_0^g \mod t^2.$$

In particular, $\underbrace{g^*(g^*\cdots(g^*))\cdots)}_{o(g)\text{-times}} t = t = t(u_0^g)^{o(g)} \bmod t^2$

So

- $\left(u_0^g(z_2,\dots z_n)\right)^{o(g)} = 1 \Rightarrow u_0^g(z_2,\dots z_n) = \text{const} = u_0^g$;
- $u_0^g \in S^1 \subset \mathbb{C}^*$ is a root of unity;
- $\tilde{\chi}: \operatorname{In}(T) \longrightarrow \mathbb{C}^*$, $h \longmapsto u_0^h$ is a group homomorphism

Let $g \in \operatorname{Ker}(\tilde{\chi})$, so $u_0^g = 1 \Rightarrow$

This means $g^* t = t + O(t^2)$; let us write

$$g^* t = t + v t^s + O(t^{s+1}) \text{ with } v = v(z_1, \dots, z_n).$$

Then mod t^{s+1} we have:

$$(g^2)^* t = g^*(g^* t) = g^*(t + v t^s) = g^* t + v\, g^*(t^s)$$
$$= t + v t^s + v t^s$$
$$= t + 2v t^s + O(t^{s+1})$$

$$\Rightarrow t = (g^{o(g)})^* t = t + o(g) v t^s + O(t^{s+1}) \Rightarrow o(g) v = 0 \underset{(\text{on } \mathbb{C})}{\Rightarrow} v = 0$$

$\Rightarrow g^* t = t.$

With the same argument, we have $g^* z_s = z_s$:

$g^* z_s = z_s + t^s v$

mod t^{s+1}: $(g^2)^* z_s = g^*(g^* z_s) = g^*(z_s + t^s v)$

$$= g^*(z_s) + t^s v = z_s + t^s v + t^s v = z_s + 2v t^s$$

$\Rightarrow o(g) v = 0 \Rightarrow v = 0 \Rightarrow g^* z_s = z_s.$

Thus, $g = \mathrm{Id}$ on $\mathcal{U}$ $\Rightarrow$ $g = \mathrm{Id}$ on X.

This means $\tilde{\chi} : \mathrm{In}(T) \to \mathbb{C}^*$ is injective, namely $\mathrm{In}(T) \leq \mathbb{S}^1$ is a subgroup of $\mathbb{S}^1$

$\Rightarrow$ $\mathrm{In}(T)$ is cyclic, that $\tilde{\chi}$ is a character of $\mathrm{In}(T)$, with order $|\mathrm{In}(T)|$, so it generates $\mathrm{In}(T)^*$.

Let us prove 2) point (b).

Let us consider the variable z_J:

We choose $z_J' := \frac{1}{|\mathrm{In}(T)|} \sum_{g \in \mathrm{In}(T)} g^* z_J$

Then $\forall h \in \mathrm{In}(T)$ we have

$$h^* z_J' = \frac{1}{|\mathrm{In}(T)|} \sum (hg)^* z_J = z_J'$$

Furthermore, we have $z_J' = z_J \mod t$ by construction.

Now let us consider the variable t:

We define $t_{\tilde{\chi}} := \frac{1}{|\mathrm{In}(T)|} \cdot \sum_{g \in \mathrm{In}(T)} \overline{\tilde{\chi}(g)}\, g^* t$

By construction, $h^* t_{\tilde{\chi}} = \frac{1}{|\mathrm{In}(T)|} \sum_{g \in \mathrm{In}(T)} \overline{\tilde{\chi}(g)}\, h^* g^* t$

$$= \frac{1}{|\mathrm{In}(T)|} \sum_{g \in \mathrm{In}(T)} \tilde{\chi}(h) \overline{\tilde{\chi}(gh)}\, h^* g^* t$$

$$= \hat{\chi}(h) \cdot t_{\tilde{\chi}} .$$

Furthermore, we have $t_{\tilde{\chi}} = t \mod t^2$. Indeed

$$t_{\tilde{\chi}} = \frac{1}{|\mathrm{In}(T)|} \sum_{g \in \mathrm{In}(T)} \overline{\tilde{\chi}(g)}\, \underbrace{g^* t}_{\tilde{\chi}(g) t \mod t^2} = \frac{1}{|\mathrm{In}(T)|} \cdot |\mathrm{In}(T)| \cdot t = t \mod t^2$$

Finally, we need to prove $(t, z_2, \ldots z_n) \xmapsto{\Psi} (t_{\tilde{x}}, z_2', \ldots z_n')$ is a new parametrization of $\mathcal{U}$. To do this, it is sufficient to prove $J\Psi$ has max. rank n at p, by Inverse Function Theorem. However, by construction we have:

$$\nabla t_{\tilde{x}} = (1 + 2t \cdot \text{something},\ 0 + t^2 \cdot \text{something},\ 0 + t^2 \text{something}, \ldots)$$

remind $t_{\tilde{x}} = t \bmod t^2$

$$\nabla z_2' = (\text{something} + 2t \cdot \text{something},\ 1 + t\,\text{something},\ 0 + t\,\text{something},\ \ldots\ldots)$$

remind $z_2' = z_2 \bmod t$

$\vdots$

$$\nabla z_n' = (\text{something} + 2t(\text{something}),\ 0 + t \cdot \text{something},\ \ldots\ldots,\ 1 + t\,\text{someth.})$$

$z_n' = z_n \bmod t$

Thus, if we evaluate the Jacobian Matrix at $p \in T = (t=0)$ we have

$$J\Psi(p) = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ * & 1 & 0 & \cdots & 0 \\ * & 0 & 1 & 0 \cdots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ * & 0 & \cdots & 0 & 1 \end{pmatrix} \Rightarrow \det J\Psi(p) = 1 \neq 0$$

$\Rightarrow \Psi$ is a new parametrization of $\mathcal{U}$.

We have then constructed local coordinates of $\mathcal{U}$ s.t. $\forall g \in In(T)$, $(t_{\tilde{x}}, z_2', \ldots z_n') \xmapsto{g} (\tilde{\chi}(g) t_{\tilde{x}}, z_2', \ldots z_n')$.

It is clear from the construction that $\tilde{\chi}$ does not

depend on the choice of local coord. at p.
Furthermore, $\tilde{x}(g) = u_0^g$ is constant on $\mathcal{U}$, so we have constructed a locally constant map to a finite set:

$$\begin{aligned} T^\circ &\longrightarrow \operatorname{In}(T)^* \\ p &\longmapsto \tilde{x} \end{aligned}$$

However, T° is a open connected Zariski subset, so the map is constant $\Rightarrow$ $\tilde{x}$ depends only on T and not on the specific point $p \in T^\circ$.
This proves 2)(c).
Let us prove point 2)(d). The map π decomposes locally as follows:

$$\begin{aligned} \pi_{|\mathcal{U}} : \mathcal{U} &\xrightarrow{\pi'} \mathcal{U}/_{\operatorname{In}(T)} \xrightarrow{f} \pi(\mathcal{U}) \subseteq Y \\ x &\longmapsto \operatorname{In}(T)\cdot x \longmapsto G\cdot x \end{aligned}$$

We observe that f is an isomorphism since if $Gx = Gy$ with $x, y \in \mathcal{U}$ $\Rightarrow$ $y = g\cdot x$ for some $g \in G$.
However, by construction of $\mathcal{U}$, then $y = g\cdot x$ with $y, x \in \mathcal{U}$ implies $g \in \operatorname{In}(T)$, so f is injective.
Moreover, we observe that the action of $\operatorname{In}(T)$ on $\mathbb{C}[t_{\tilde{x}}, z_2, \dots z_n]$ gives the invariant subring

$$\mathbb{C}[t_{\tilde{x}}, z_2, \dots z_n]^{\operatorname{In}(T)} = \mathbb{C}[t_{\tilde{x}}^{|\operatorname{In}(T)|}, z_2, \dots z_n],$$

so $\mathcal{U}/\mathrm{In}(T)$ is smooth and the quotient map is
$$(t_{\tilde{x}}, z_2, \dots z_n) \longmapsto (t_{\tilde{x}}^{|\mathrm{In}(T)|}, z_2, \dots z_n).$$
Finally, point (3) follows directly from point 2)(d) ▨

Let T be an irreducible component of R.
Then $\mathrm{In}(T)$ is cyclic
$\langle \tilde{x} \rangle = \mathrm{In}(T)^*$, in particular there exists
a unique $g \in \mathrm{In}(T)$ such that $\tilde{x}(g) = e^{\frac{2\pi i}{|\mathrm{In}(T)|}} =: \xi$
is the first $|\mathrm{In}(T)|$-root of the unity.

Def The *local monodromy* of a irreducible component T of R for $\pi: X \to Y$ is the unique element $g \in G$ such that g acts locally around (a gen. point of) T as the multiplication of the first root of the unity:
$$(t, z_2, \dots z_n) \xrightarrow{\ g\ } (\xi \cdot t, z_2, \dots z_n), \quad T = (t=0)$$

To any irreducible component T of R we can attach its local monodromy $g \in G$, so we can set
$T_g := T$.
Let R_g be the sum of the irreducible components of R sharing the same local monodromy g: $R_g := \sum_i T_g$.

Let us consider two components T_1 and T_2 belonging to the same orbit, so $T_2 = h \cdot T_1$ for some $h \in G$.
Then if g is the local monodromy of T_1, hgh^{-1} is the local modromy of T_2.
Thus, the local monodromies of components belonging to the same orbit are conjugated to each other.

Def If G is an $\boxed{\text{abelian group}}$, then $hgh^{-1} = g$, so T_2 and T_1 have the same local monodromy; in other words, R_g is a sum of orbits. We denote by Δ_g
An irreducible component Δ of the reduced branch divisor D is denoted as $\Delta = \Delta_g$ if $\pi^{-1}(\Delta_g)$ consists of components with the same local monodromy $g \in G$.
We denote by $D_g := \sum'_{\Delta \leq D, \Delta = \Delta_g} \Delta_g$.

Thus, we have the following decomposition of the reduced ramification and branch divisor of $\pi : X \to Y$:

$$R = \sum_{g \in G}' R_g \qquad D = \sum_{g \in G}' D_g \quad .$$

We have constructed a set of divisors of Y labeled by the elements of G: $\{D_g\}_{g \in G}$.

Remark By construction of D_f and R_f and by the previous theorem, we have

$$\boxed{\pi^* D_f = o(f) \cdot R_f}$$